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Covariant Kolmogorov equation and entropy current for the relativistic Ornstein-Uhlenbeck process

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Abstract. The relativistic Ornstein-Uhlenbeck Process (ROUP), which is a toy-model of relativistic irreversible phenomena, is studied statistically in an explicitly covariant manner. An 8-dimensional phase space is introduced (four dimensions for space-time coordinates, and four dimensions for the 4-momentum coordinates), on which 'extended' probability distributions are defined (the usual probability distribution is recovered as their restriction to the mass shell). An explicitly covariant Kolmogorov equation is derived for these 'extended' probability distributions. The whole formalism is used to introduce a 4-current of conditional entropy and prove that the 4-divergence of this 4-current is always positive. This constitutes an H-theorem for the ROUP.

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1 Introduction

In 1997, Debbasch *et al.* introduced the Relativistic Ornstein-Uhlenbeck process (ROUP) as a toy-model of special relativistic irreversible phenomena. Its Galilean counterpart, the usual Ornstein-Uhlenbeck process, models the motion of a particle which diffuses in a given fluid by a couple of stochastic equations which govern the time-evolution of both the position and velocity of the particle. If the fluid in which the particle diffuses is in a thermodynamical equilibrium, the Galilean Ornstein-Uhlenbeck process has a natural preferred inertial frame, which is the global rest-frame of the fluid. Indeed, the study of the Galilean process is usually carried out in that frame only, without explicit mention that the process could also be investigated in other reference frames.

The ROUP naturally shares some features with its Galilean predecessor. For physical reasons which have already been discussed at length in [1], the ROUP was not built to be considered as a realistic model of Relativistic diffusion. Nevertheless, it can still be described, mainly for simplicity reasons, by using the image of a Brownian special relativistic particle diffusing in a surrounding (special relativistic) fluid. Let us suppose that this fluid is in a state of equilibrium. Then, there exists an inertial frame $(\mathcal{R})$ where the fluid is at rest; this frame is usually called the global rest-frame of the fluid; it then constitutes a naturally preferred inertial frame for the ROUP. The ROUP has been originally defined by the couple of covariant stochastic evolution equations which, in theory, permit the study of the ROUP in an arbitrary inertial frame. As in the Galilean case, these equations determine the time-evolution of both position and momentum of the diffusing particle. It is to be noted, however, that, by construction, the noise used in these equations takes a simple, easily tractable form in $(\mathcal{R})$ only.

In the Galilean problem, an important consequence of the equations of motion of the Brownian particles is the transport equation verified by the distribution function in phase-space. This equation is commonly called the Kramers or forward Kolmogorov equation. In the relativistic case, this transport equation turns out to be even more important than in the Galilean problem because the original stochastic equations which define the ROUP are nonlinear (in contradistinction to their Galilean counterparts) and the relativistic process can only be practically studied through its associated transport equation. The relativistic Kramers equation has been first obtained in $(\mathcal{R})$, the preferred frame of the process and only later on in an arbitrary inertial frame [2]. However, because of the technique used in the derivation, this equation has been written so far in a covariant but not manifestly covariant fashion. In particular, the phase-space has been considered to be the physical 6-dimensional one and the distribution function of the diffusing particle, in each frame, has been written as a function of time in that frame and of the coordinates of the particle in this usual 6-dimensional

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phase-space. In relativistic statistical physics, it has become customary to introduce an extended 8-dimensional phase-space, which is essentially the (Cartesian) product of the space-time manifold and of a corresponding extended 4-dimensional momentum-space (more precisely, the extended phase-space can be identified with the tangent bundle to the space-time manifold). A distribution function is then introduced on this extended phase-space, treating the fourth momentum-component as an independent variable. Every calculation is then carried out with this distribution function and the physical results can be recovered by restricting every equation to the mass-shell. This sort of formalism is usually considered very elegant and, because it really treats time as a space-time independent coordinate with its associated independent momentum coordinate, it generally simplifies calculations a lot. Moreover, the use of a manifestly covariant formalism seems mandatory in the context of general relativity.

The aim of the present article is to introduce such a manifestly covariant formalism for the ROUP and to propose a first application of this formalism. More precisely, this article is organized as follows. In Section 2, we first review some basic facts about the ROUP and introduce the principal tools which will be of use in the other sections. In particular, the basics of a manifestly covariant formalism adapted to the ROUP are introduced at this stage. This formalism differs from the most usual, manifestly covariant formalism commonly used in relativistic kinetic theory; indeed, for the transport equation associated to the ROUP to have regular coefficients over the whole extended phase-space, it is necessary to limit the momentum-space to half the 4-dimensional space $\mathbb{R}^4$ (usual manifestly covariant kinetic theory uses the whole space $\mathbb{R}^4$ as momentum-space). For various technical reasons, it is then *not* obvious that the distribution function defined over this extended phase-space can be chosen to be a Lorentz-scalar, as is customarily done in the more standard manifestly covariant formalism. We have therefore included in Appendix A a detailed proof of this important result. In Section 3, we derive for the ROUP the manifestly covariant transport equation verified by the distribution function defined on the extended 8-dimensional phase-space. As expected, this equation turns out to be simpler than the original Kramers equation verified by the standard distribution function defined on the usual 6-dimensional phase-space. In Section 4, we use the formalism just developed to show that a conditional entropy 4-current can be associated to the ROUP and we also prove that the 4-divergence of this 4-current is always non-negative. This proves that a frame-dependent conditional entropy can be associated to the ROUP and that this entropy is a never decreasing quantity, as it should naturally be. Some calculations necessary to prove this H-theorem are actually quite heavy and their detailed presentation has therefore been relegated to Appendix B. Finally, we review our results in Section 5 and discuss their possible extensions.

2 Fundamentals

2.1 Notation

In this article, c denotes the velocity of light, and the signature of the metric is chosen to be $(+,-,-,-)$. Boldfaced symbols designate 3-vectors, whereas 4-vectors appear as normal italic symbols. m and γ are respectively the mass of the "diffusing" particle and its Lorentz factor, whose expression in terms of the 3-momentum $\mathbf{p}$ of the particle reads: $\gamma(\mathbf{p}) = \sqrt{1 + \frac{\mathbf{p}^2}{m^2c^2}}$. T designates the absolute temperature and k the Boltzmann constant. As usual, Greek indices label the components of 4-vectors and run from 0 to 3, whereas Latin ones label the components of 3-vectors and run from 1 to 3.

2.2 The ROUP

In $(\mathcal{R})$, the global rest-frame of the fluid which surrounds the diffusing particles, the ROUP can be defined by the following couple of stochastic equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{x} = \dfrac{\mathbf{p}}{m\gamma} \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{p} = -\alpha\,\dfrac{\mathbf{p}}{\gamma} + \sqrt{2D}\,\dfrac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} \end{cases}, \tag{1}$$

where $\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t}$ indicates that the stochastic part $\mathbf{F}_s$ of the force which acts on the diffusing particle is, up to the multiplicative constant $\sqrt{2D}$, the derivative of the Wiener process, *i.e.*, a Gaussian white noise. The positive constant α enters the definition of the deterministic part $\mathbf{F}_d$ of the force acting on the particle and plays the role of a friction coefficient, as in the Galilean case.

From these equations, it is possible to derive a transport equation which fixes, in the same reference frame, the time-evolution of the distribution function $\Pi(t, \mathbf{x}, \mathbf{p})$, defined over the standard phase space $\mathbb{R}^6$ and associated to the usual measure $\mathrm{d}^3x\mathrm{d}^3p$ (see [1]):

$$\partial_t \Pi + \nabla_{\mathbf{x}}\,.\,\left(\frac{\mathbf{p}}{m\gamma}\,\Pi\right) + \nabla_{\mathbf{p}}\,.\,\left(-\alpha\frac{\mathbf{p}}{\gamma}\,\Pi\right) = D\,\Delta_{\mathbf{p}}\Pi. \tag{2}$$

As discussed in [2], there are essentially two distinct ways of obtaining the transport equation in another inertial frame. The first one is simply to invoke the scalar nature of the relativistic distribution function Π. The other one is based directly on stochastic calculus and, being much more involved than the first one, cannot be summed up here. As they should, both methods deliver the same result. If one introduces another inertial frame $(\mathcal{R}')$, which moves with velocity $\mathbf{V} = V\mathbf{e}_x$ with respect to $(\mathcal{R})$ and whose origin and axes coincides with those of $(\mathcal{R})$ at time $t = 0$, one has, with obvious notations:

$$\partial_{t'}\Pi' + \nabla_{\mathbf{x}'}(\frac{\mathbf{p}'}{m\gamma'}\Pi') + \nabla_{\mathbf{p}'}(\mathbf{F}'_d\Pi') =$$

$$\begin{aligned} & D\,\frac{\partial}{\partial p'^x}\left\{\frac{1}{d'}\left(1+\ \frac{\Gamma^2\beta^2(p'^{y2}+p'^{z2})}{p'^2_0}\right)\frac{\partial \Pi'}{\partial p'^x}\right\} \\ & -D\,\frac{\partial}{\partial p'^x}\left\{\Gamma\beta\frac{p'^y}{p'_0}\,\frac{\partial \Pi'}{\partial p'^y} - \Gamma\beta\frac{p'^z}{p'_0}\,\frac{\partial \Pi'}{\partial p'^z}\right\} \\ & +D\,\frac{\partial}{\partial p'^y}\left\{-\Gamma\,\beta\frac{p'^y}{p'_0}\,\frac{\partial \Pi'}{\partial p'^x} + d'\,\frac{\partial \Pi'}{\partial p'^y}\right\} \\ & +D\,\frac{\partial}{\partial p'^z}\left\{-\Gamma\,\beta\frac{p'^z}{p'_0}\,\frac{\partial \Pi'}{\partial p'^x} + d'\,\frac{\partial \Pi'}{\partial p'^z}\right\}. \end{aligned} \tag{3}$$

In the above equation, β stands for V/c, Γ for $\frac{1}{\sqrt{(1-\beta^2)}}$, and d' for the quantity $\Gamma\left(1+\frac{p'^x}{p'_0}\right)$. Actually, (3) is formally a new result; indeed, only the spatially one-dimensional, simpler version of (3) has already been published [2]. However, equation (3) can be obtained by exactly the same method as the one introduced in [1] and we feel that a complete presentation of that rather heavy calculation is not necessary here.

It is evident that (3) as it stands *cannot* be used for any practical calculations. As hinted to in the Introduction to this article, a natural way to remedy the situation is to introduce a manifestly covariant formalism in which the four momentum components will be treated as independent variables. The physical results will then be recovered by restricting every equation to the mass-shell.

2.3 Manifestly covariant formalism

Let us begin by introducing the relativistic extended phase-space for the Brownian particle. In any inertial frame $(\mathcal{S})$, this phase-space is 8-dimensional; the first four degrees of freedom are the four space-time coordinates in that frame and the remaining ones are the associated four momentum components. The three spatial momentum-components can naturally take any real value. However, the range of variation one should choose for the zeroth momentum-component treated as an independent degree of freedom is not obvious. Many authors seem to implicitly retain the whole real-axis but, for reasons which will be made clear in Section 3.2, such a choice is not advisable if one wants to develop a manifestly covariant formalism for the ROUP. If one conventionally denotes by $\mathcal{P}$ the 4-dimensional region of $\mathbb{R}^4$ to which the variation of the 4-vector p is restrained, it will be argued in Section 3.2 that a natural choice for $\mathcal{P}$ is the "half-space" defined by the condition: $pU > 0$, where U stands for the 4-velocity of the fluid in which the Brownian particles diffuse. We will show that this choice for $\mathcal{P}$ is the simplest possible one which ensures that every coefficient in the manifestly covariant transport equation that will be obtained in the next section is actually regular on the whole phase-space.

In any given reference frame, the condition $pU > 0$ can be transcribed in terms of the the zeroth component of p and reads:

$$p^0 > \epsilon(U, \mathbf{p}) \tag{4}$$

with the quantity $\epsilon(U, \mathbf{p})$ defined by:

$$\epsilon(U, \mathbf{p}) = \frac{\mathbf{pU}}{U^0}\,. \tag{5}$$

Reasonably enough, the mass-shell is therefore included in P; indeed, the sign of the scalar pU on the mass-shell can be checked by evaluating this quantity in $(\mathcal{R})$, the inertial (Lorentz-)frame in which the 3-velocity $\mathbf{U}$ of the fluid which surrounds the diffusing particles vanishes; if $\mathbf{p}$ is the 3-momentum of the Brownian particle in that frame, one can write trivially $pU = mc\gamma(\mathbf{p})$; this makes clear that the mass-shell is included in $\mathcal{P}$.

To any function g defined on the extended phase-space corresponds a unique, time-dependent function $\tilde{g}$ on the physical 6-dimensional phase-space:

$$\tilde{g}(t, \mathbf{x}, \mathbf{p}) = \int_{\epsilon(U,\mathbf{p})}^{+\infty} g(t, \mathbf{x}, p^0, \mathbf{p})\,\delta\Big(p^0 - mc\gamma(\mathbf{p})\Big)\;\mathrm{d}p^0. \tag{6}$$

It is therefore natural, *in any reference-frame*, to introduce a function f defined over the extended phase-space, such that $\tilde{f}$ is identical to the usual distribution function Π defined over the 6-dimensional physical phase-space:

$$\Pi(t, \mathbf{x}, \mathbf{p}) = \int_{\epsilon(U,\mathbf{p})}^{+\infty} f(t, \mathbf{x}, p^0, \mathbf{p})\,\delta\Big(p^0 - mc\gamma(\mathbf{p})\Big)\;\mathrm{d}p^0. \tag{7}$$

Naturally, given a distribution Π, it is always possible to find at least one f which verifies (7) and the solution is generally *not* unique.

In conventional manifestly covariant kinetic theory, it is a standard result that the distribution function f defined on the extended phase-space $\mathbb{R}^4 \times \mathbb{R}^4$ is a Lorentz-scalar. That f is also a Lorentz-scalar in the present formalism is not obvious at all, notably because the zeroth momentum-component has now a restricted range of variation. We therefore felt necessary to present in this article a complete proof that, in the manifestly covariant formalism associated to the ROUP, f can also be chosen as a Lorentz-scalar. Because this proof is rather intricate, it is proposed as an appendix to the main text.

3 Manifestly covariant Kolmogorov equation

The final aim of this section is to obtain a transport equation verified by the distribution function f on the extended phase-space which, after restriction to the mass-shell, gives back Kramers equation for the original distribution Π on the physical 6-dimensional phase-space. The simplest way to proceed is to work first in the preferred inertial frame of the ROUP, $(\mathcal{R})$, where Kramers equation for Π takes a simpler form, and then to boost the obtained transport equation for f in this frame to another arbitrary inertial frame.

3.1 Transport equation on the extended phase-space in $(\mathcal{R})$

By definition, in $(\mathcal{R})$, the 4-velocity U of the fluid in which the particles diffuse has vanishing spatial components and its time component is therefore equal to unity (see also below). In $(\mathcal{R})$, the region $\mathcal{P}$ accessible to the 4-momentum p is therefore only restricted by the condition: $p^0 > 0$. Let now h be any function defined over the extended phase-space in $(\mathcal{R})$ and let $\tilde{h}$ be its restriction to the mass-shell, defined by a relation similar to (6). The well-known properties of the delta-function imply that:

$$\nabla_{\mathbf{p}} \tilde{h} = \int_0^{+\infty} \mathbf{C}(h)\, \delta\Big(p^0 - mc\gamma(\mathbf{p})\Big)\, \mathrm{d}p^0, \tag{8}$$

where the differential operator $\mathbf{C}$ is defined by:

$$\mathbf{C} = \frac{1}{p^0}(\mathbf{p}\ \partial_{p^0} + p^0\ \mathrm{d}ppv). \tag{9}$$

In that way, it becomes possible to re-express in (2) every derivative of Π with respect to $\mathbf{p}$ in terms of derivatives of f. On thus obtains the following equation, which is mathematically equivalent to (2):

$$\int_0^{+\infty} \frac{1}{p^0} \mathcal{L}(f)\ \delta\Big(p^0 - mc\gamma(\mathbf{p})\Big)\, \mathrm{d}p^0 = 0 \tag{10}$$

where the differential operator $\mathcal{L}$ is defined by:

$$\mathcal{L}(f) = p^0 \partial_t f + \mathbf{p} \cdot \nabla_{\mathbf{x}} f + p^0 \mathbf{C}(\mathbf{F}_d f) - p^0 D\ \mathbf{C}^2(f). \tag{11}$$

As before, $\mathbf{F}_d$ stands for the deterministic 3-force acting on the diffusing particles. According to (1):

$$\mathbf{F}_d = -\alpha\ \frac{\mathbf{p}}{\gamma(\mathbf{p})}\ \cdot \tag{12}$$

In equation (11), one can actually use for $\mathbf{F}_d$ any other expression which reduces to (12) on the mass-shell; this point will be discussed with greater detail in the immediately following section. A sufficient (but not necessary) condition for equation (2) to be verified is simply that f belongs to the kernel of $\mathcal{L}$:

$$\mathcal{L}(f) = 0. \tag{13}$$

Equation (13) is the desired Kolmogorov equation for f in $(\mathcal{R})$.

3.2 Manifestly covariant form of Kolmogorov equation

To obtain a manifestly covariant Kolmogorov equation for f, one has to express the operator $\mathcal{L}$ in a manifestly covariant manner. Since f can be chosen as a Lorentz-scalar, one expects $\mathcal{L}$ also to be a Lorentz-scalar. The first two terms on the right-hand side of (11) can trivially be rewritten as $p^\mu \partial_{x^\mu} f$. The following two terms involve the deterministic 3-force $\mathbf{F}_d$. To express their contribution in a manifestly covariant manner, it is only logical to revert to the manifestly covariant expression for the deterministic force introduced in [1]:

$$F_d^\mu = -mc\lambda^\mu_\nu \left(u^\nu - U^\nu\right) + mc\lambda^\alpha_\beta u_\alpha \left(u^\beta - U^\beta\right)\ u^\mu \tag{14}$$

where, as before, U stands for the 4-velocity of the fluid in which the particle diffuses and u stands for the velocity 4-vector of the diffusing particle itself. The tensor λ characterizes the 'friction' of the fluid on the particle. Its physically correct expression has also been given in [1] and reads:

$$\lambda^\mu_\nu = \frac{\alpha}{(uU)^2}\, \Delta^\mu_\nu \tag{15}$$

where Δ is the 4-dimensional projector on the hypersurface orthogonal to U, the 4-velocity of the fluid which surrounds the Brownian particle. Because λ is proportional to the projector Δ, equation (14) actually simplifies into:

$$F_d^\mu = -mc\lambda^\mu_\nu u^\nu + mc\lambda^\alpha_\beta u_\alpha u^\beta\ u^\mu. \tag{16}$$

Equation (16) is actually valid on the mass-shell and is susceptible of various off-shell generalizations. A convenient definition of the deterministic 4-force acting on the particle in terms of its possibly off-shell momentum p is:

$$F_d^\mu = -\lambda^\mu_\nu p^\nu\ \frac{p^2}{m^2c^2} + \lambda^\alpha_\beta\ \frac{p_\alpha p^\beta}{m^2c^2}\ p^\mu, \tag{17}$$

with the tensor λ related to p by the relation:

$$\lambda^\mu_\nu = \frac{\alpha(mc)^2}{(pU)^2}\, \Delta^\mu_\nu. \tag{18}$$

First, (17) clearly reduces on the mass-shell to (16). Second, definition (17) presents the advantage of making F_d orthogonal to p, even off-shell. One can introduce a 3-vector $\mathbf{F}_d$, defined by:

$$F_d = \frac{1}{mc}\left(\mathbf{p} \cdot \mathbf{F}_d, p^0 \mathbf{F}_d\right), \tag{19}$$

and definition (14) then leads to the following expression for $\mathbf{F}_d$ in $(\mathcal{R})$:

$$\mathbf{F}_d = -\alpha mc\, \frac{\mathbf{p}}{p^0}, \tag{20}$$

which does reduce to (12) when p is on the mass-shell. It is then straightforward to check that the contribution of the two terms involving $\mathbf{F}_d$ in (11) can be simply written as $mc\partial_{p^\mu}(F_d^\mu f)$.

The first step in dealing with the last contribution to $\mathcal{L}$ is to view the differential operator $\mathbf{C}$ as the 3-dimensional vectorial part of a 4-dimensional operator C^μ whose time-component vanishes in $(\mathcal{R})$. Considering that, in $(\mathcal{R})$, the components of U and those of the projector Δ are given by:

$$U^\mu = (1, 0, 0, 0), \tag{21}$$

and

$$\Delta^\mu_\nu = \begin{pmatrix} 0\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix}, \tag{22}$$

it is straightforward to verify that:

$$C^i = \left(U^\beta \Delta^{i\nu} - U^\nu \Delta^{i\beta}\right) \frac{p_\beta}{pU} \frac{\partial}{\partial p^\nu} \tag{23}$$

and that the manifestly covariant expression for C is:

$$C^\mu = \left(U^\beta \Delta^{\mu\nu} - U^\nu \Delta^{\mu\beta}\right) \frac{p_\beta}{pU} \frac{\partial}{\partial p^\nu} \cdot \tag{24}$$

The last term on the right-hand side of (11) can therefore be identified as the form taken by the scalar operator $-DC_\mu\, C^\mu$ in the reference frame $(\mathcal{R})$.

Adding up all the preceding contributions, one obtains the manifestly covariant form of $\mathcal{L}$:

$$\mathcal{L}(f) = p^\mu \partial_{x^\mu} f + mc\, \partial_{p^\mu}(F^\mu_d f) - (pU)\ DC_\mu\ C^\mu(f). \tag{25}$$

One can make explicit the differential nature of C by introducing the tensor K, whose contravariant components are defined by:

$$\begin{aligned} K^{\mu\rho\beta\nu} = \quad & U^\mu U^\beta \Delta^{\rho\nu} - U^\mu U^\nu \Delta^{\rho\beta} \\ & + U^\rho U^\nu \Delta^{\mu\beta} - U^\rho U^\beta \Delta^{\mu\nu}. \end{aligned} \tag{26}$$

The manifestly covariant Kramers equation then reads:

$$p^\mu \partial_{x^\mu} f + \partial_{p^\mu}\left(mcF^\mu_d f\right) - DK^{\mu\rho\beta\nu}\ \partial_{p^\rho}\left(\frac{p_\mu\, p_\beta}{pU}\ \partial_{p^\nu} f\right) = 0. \tag{27}$$

To facilitate further manipulations, it is also convenient to group all terms containing only first derivatives with respect to the various components of p. Equation (27) can be rewritten as:

$$p^\mu \partial_{x^\mu} f + \partial_{p^\mu}\left(I^\mu f\right) - \partial_{p^\mu p^\nu}\left(J^{\mu\nu} f\right) = 0, \tag{28}$$

where the tensors I and J are defined by:

$$I^\mu = mcF^\mu_d + DK^{\rho\mu\beta\nu} \partial_{p^\nu}\left(\frac{p_\rho p_\beta}{pU}\right) \tag{29}$$

and

$$J^{\mu\nu} = DK^{\rho\mu\beta\nu}\ \frac{p_\rho p_\beta}{pU} \cdot \tag{30}$$

Time has now come to discuss and justify our choice for the region $\mathcal{P}$ accessible to the 4-momentum p of the particle. Equations (29) and (30) make clear that some coefficients in the manifestly covariant transport equation become singular on the hypersurface $pU = 0$. We have therefore chosen for $\mathcal{P}$ the largest region in momentum-space which contains the mass-shell and in which the coefficients of the manifestly covariant transport equation exhibit no singularity.

4 Entropy 4-current

4.1 Definition of the entropy current

In the general theory of Markovian stochastic processes, it is customary to introduce the concept of conditional entropy [3]. Let X be a set of stochastic variables whose time-evolution in some (phase-)space Ω is governed by a given Markovian process. Let now f and g be two probability distributions defined on Ω. The conditional entropy $H_c(f \mid g)$ of f with respect to g can be conveniently defined by [3]:

$$H_c(f \mid g) = -\int_\Omega f\left[\ln\left(\frac{f}{g}\right) + \frac{g}{f} - 1\right] dX, \tag{31}$$

where dX stands for the (usual) Lebesgue measure on Ω. The conditional entropy thus defined is susceptible of at least two different physical interpretations. We refer the reader to [3] for a substantial general discussion of the issue. In any case, if one chooses for g an invariant measure of the process, *i.e.*, a stationary solution of the transport equation, and computes the conditional entropy of an arbitrary solution f of the transport equation at various times, one obtains a time-dependent quantity $\tilde{H}_c(t)$ and a theorem due to Voigt [3] proves that, under rather general circumstances, $\tilde{H}_c(t)$ is a non-decreasing function of time, which justifies the name entropy.

To follow the traditional relativistic theories of continuous media, one may wish to introduce not only an entropy for the ROUP but also an entropy 4-current. It is easier to proceed in a manifestly covariant manner and this is where the formalism introduced in the preceding sections proves most useful. The first step consists in identifying a time- and position- independent solution of (27) which can serve as stationary equilibrium distribution in phase-space. It is relatively straightforward to check that the following distribution f^* qualifies:

$$f^*(p) = \frac{1}{4\pi(mc)^3} \frac{\frac{mc^2}{kT}}{K_2(\frac{mc^2}{kT})}\, e^{-\frac{c}{kT}(pU)}, \tag{32}$$

where K_2 is the second order modified Hankel function, k is the Boltzmann constant, and T the temperature of the surrounding fluid. As may have been expected, f^* is simply the standard Maxwell-Jüttner distribution [4] written in a manifestly covariant manner (see for example [5]).

A candidate for the entropy 4-current S^μ_c should be constructed from the expression $H_c(f \mid g)$ as the usual particle 4-current j^μ is constructed from the total number N of particles. It is customary in manifestly covariant relativistic kinetic theory to define the particle current by the expression [6]:

$$j^\mu(x) = 2\int_{\mathbb{R}^4} p^\mu f(x,p)\theta(p^0)\delta\left(p^2 - m^2c^2\right) d^4p, \tag{33}$$

where the θ function enforces the positivity of the time component of p in the integral. To link this definition with the developments presented earlier in Section 2.3, one can use the identity:

$$\delta(p^2 - m^2c^2) = \frac{1}{2p^0}\left[\delta\left(p^0 - mc\gamma(\mathbf{p})\right) + \delta\left(p^0 + mc\gamma(\mathbf{p})\right)\right]. \tag{34}$$

Equation (33) then delivers:

$$n(x) = \int_{\mathbb{R}^4} f(x,p)\delta\left(p^0 - mc\gamma(\mathbf{p})\right) \mathrm{d}^4 p \tag{35}$$

and

$$\mathbf{j}(x) = \int_{\mathbb{R}^4} \frac{\mathbf{p}}{p^0} f(x,p)\delta\left(p^0 - mc\gamma(\mathbf{p})\right) \mathrm{d}^4 p, \tag{36}$$

where, as usual, n and $\mathbf{j}$ stand respectively for the spatial particle density and its associated 3-current. Both expressions are clearly consistent with the definition of Π in terms of f given by equation (7). Considering (31) and (33), the most natural definition for the entropy 4-current S_{c}^{μ} reads [6,7]:

$$S_{\mathrm{c}}^{\mu}(x) = -\int_{\mathcal{P}} p^{\mu} f \left[\ln\left(\frac{f}{f^*}\right) + \frac{f^*}{f} - 1\right] \times \theta(p^0)\delta(p^2 - m^2c^2)\ \mathrm{d}^4 p. \tag{37}$$

4.2 H-theorem

Let us now prove that the 4-divergence of the conditional entropy S_{c}^{μ} is indeed non-negative. Since f^* does not depend on x, direct differentiation of (37) leads to:

$$\partial_{x^{\mu}} S_{\mathrm{c}}^{\mu} = -\int_{\mathcal{P}} (p^{\mu}\partial_{x^{\mu}} f) \ln\left(\frac{f}{f^*}\right) \times \theta(p^0)\ \delta(p^2 - m^2c^2)\ \mathrm{d}^4 p. \tag{38}$$

Some non-trivial algebra, rejected in Appendix B, is necessary to derive from (38) the following expression for $\partial_{x^{\mu}} S_{\mathrm{c}}^{\mu}$:

$$\partial_{x^{\mu}} S_{\mathrm{c}}^{\mu} = \int_{\mathcal{P}} f J^{\mu\nu}\mathcal{D}_{\mu}\mathcal{D}_{\nu}\ \theta(p^0)\ \delta(p^2 - m^2c^2)\ \mathrm{d}^4 p. \tag{39}$$

The tensor J has already been defined by equation (30).

A sufficient condition for the integral in the preceding equation to be non-negative is for the Lorentz-scalar

$$\mathcal{J}^{\mu\nu}\mathcal{D}_{\mu}\mathcal{D}_{\nu}$$

to be also non-negative. One can most easily prove this is the case by evaluating this scalar in ($\mathcal{R}$), the proper rest-frame of the fluid in which the particle diffuses. Making use of equations (30, 26, 21) and (22), one obtains the components of $\mathcal{J}$ in that frame:

$$\begin{aligned}
\mathcal{J}^{00} &= \frac{D}{p^0}\ \mathbf{p}^2\\
\mathcal{J}^{0i} &= -\frac{D}{p^0}\ p^0 p^i\\
\mathcal{J}^{i0} &= -\frac{D}{p^0}\ p^i p^0\\
\mathcal{J}^{ij} &= \frac{D}{p^0}\ (p^0)^2\delta^{ij},
\end{aligned} \tag{40}$$

where δ^{ij} stands for the usual 3-dimensional Kronecker symbol. This leads to:

$$\mathcal{J}^{\mu\nu}\mathcal{D}_{\mu}\mathcal{D}_{\nu} = \frac{D}{p^0}\left[\mathcal{D}^0\mathbf{p} - p^0\mathcal{D}\right]^2. \tag{41}$$

This proves that the integrand in (39) is non-negative in any reference frame and that the 4-divergence of the entropy current is therefore also non-negative. This constitutes the H-theorem for the ROUP. Indeed, one can then define, in an arbitrary inertial frame, the conditional entropy of the process to be the 3-dimensional volume integral of the first component of the entropy current S_{c} and the H-theorem proves that this entropy is a non-decreasing function of the time-coordinate in that reference frame. Naturally, when $f = f^*$, the 4-vector $\mathcal{D}$ vanishes identically and, because of (41), so does the divergence of the entropy current; in that case, the state of the system is spatially uniform in any reference frame and its conditional entropy density is also constant in time, as it should be.

5 Conclusion

In this article, we have developed for the special-relativistic Ornstein-Uhlenbeck process a manifestly covariant formalism similar to the one commonly used in relativistic statistical physics [6,8]. In particular, after having reviewed some basics facts about the ROUP, we have introduced a variant of the usual manifestly covariant formalism and we have derived a manifestly covariant transport equation for the ROUP. This equation has a simpler structure than its non-manifestly covariant equivalent introduced in earlier publications. The manifestly covariant formalism introduced in this article differs from the most usual one by the fact that the momentum-space of the diffusing particles is restricted to half the 4-dimensional space $\mathbb{R}^4$. This is the simplest possible choice which makes every coefficient in the transport equation singularity-free on the whole phase-space. We have also presented a complete proof that, with this choice of momentum-space, the distribution-function over the extended phase-space can still be chosen to be a Lorentz-scalar.

As an application of the whole formalism, we have proposed a manifestly covariant expression for the conditional entropy 4-current associated to the ROUP and we have used the (manifestly covariant) transport equation to prove an H-theorem for the process.

Until now, the ROUP has only been studied in the special relativistic context. The next logical step is naturally to extend the study of the process to the general relativistic realm. This appears to be a most formidable task if one starts from non-manifestly covariant special relativistic equations but the material presented in this article should make the job easier. The general relativistic transport equation and some possible astrophysical applications will be addressed in a forthcoming publication.

It is our pleasure to thank J. Gariel for many helpful discussions.

Appendix A

It has been proven in [2] that Π is a Lorentz-scalar. Starting from this result, it is possible to prove directly that f can be chosen to be also Lorentz-invariant. To this end, it is convenient to introduce another inertial frame $(\mathcal{S}')$, linked to $(\mathcal{S})$ by a proper Lorentz-transformation along the x-axis. With standard notations, one has:

$$\begin{cases} p'^0 = \Gamma \ \left(p^0 - \beta \ p^x\right) \\ p'^x = \Gamma \ \left(p^x - \beta \ p^0\right) \end{cases}, \tag{42}$$

and

$$\begin{cases} p^0 = \Gamma \ \left(p'^0 + \beta \ p'^x\right) \\ p^x = \Gamma \ \left(p'^x + \beta \ p'^0\right) \end{cases}, \tag{43}$$

with $\beta = V/c$ and $\Gamma = (1-\beta^2)^{-1/2}$. The other momentum components are invariant and, as far as the space-time degrees of freedom are concerned, y and z do not change either. On the contrary, one has the following well-known relations between (t', x') and (t, x):

$$\begin{cases} ct' = \Gamma \ (ct - \beta \ x) \\ x' = \Gamma \ (x - \beta ct) \end{cases}. \tag{44}$$

It is important to stress that, at this point, (42) is considered to be valid for any 4-momentum p, both on and off the mass-shell.

The fact that Π is a Lorentz-scalar is expressed mathematically by the following relation:

$$\Pi'(t', \mathbf{x}', \tilde{\mathbf{p}}') = \Pi(t, \mathbf{x}, \mathbf{p}) \tag{45}$$

where Π' designates the distribution function in $(\mathcal{S}')$ and where the 3-vector $\tilde{\mathbf{p}}'$ is obtained from the 3-vector $\mathbf{p}$ by applying the Lorentz-boost (42) to the on-shell 4-vector $(p^0 = mc\gamma(\mathbf{p}), \mathbf{p})$ (the reason for the "extra"-tilde in the notation will become clear shortly). Combining (7) and (45), one can write:

$$\Pi'(t', \mathbf{x}', \tilde{\mathbf{p}}') = \int_{\epsilon(U,\mathbf{p})}^{+\infty} f(t, \mathbf{x}, p^0, \mathbf{p}) \ \delta\Big(p^0 - mc\gamma(\mathbf{p})\Big) \ \mathrm{d}p^0. \tag{46}$$

To study the variance of f, it is therefore necessary to investigate how the integration range, the 'delta-function' and the infinitesimal $\mathrm{d}p^0$ which appear in the right-hand side of (46) transform under a Lorentz-boost. All this essentially amounts to changing the integration variable in (46) from p^0 to p'^0. Differentiating (42) at fixed $\mathbf{p}$, one obtains immediately:

$$\mathrm{d}p'^0 = \Gamma \mathrm{d}p^0, \tag{47}$$

As far as the delta-function is concerned, it is convenient to rewrite its argument σ as a function of p'^0, for a fixed $\mathbf{p}$ (or $\tilde{\mathbf{p}}'$, equivalently):

$$\begin{aligned} \sigma(p'^0) &\equiv p^0 - mc\gamma(\mathbf{p}) \\ &= \Gamma \ \Big(p'^0 + \beta \ p'^x\Big) - mc\gamma(\mathbf{p}). \end{aligned} \tag{48}$$

In this last equation, p'^x itself has to be considered as a function of the variable p'^0, for a fixed $\mathbf{p}$. This might seem strange but, on closer look, $\mathbf{p}'$ is defined by (42) and is therefore not generally identical to $\tilde{\mathbf{p}}'$, since p^0 is treated as an independent momentum coordinate, which is not required, at this stage, to be equal to $mc\gamma(\mathbf{p})$. The first equation in (42), combined with the first equation in (43) leads to the following explicit expression for p'^x in terms of p'^0 (for a fixed $\mathbf{p}$):

$$p'^x = \frac{1}{\Gamma} p^x - \beta \ p'^0. \tag{49}$$

Inserting this result in (48) yields the following explicit expression for the argument σ of the delta-function in (46):

$$\begin{aligned} \sigma(p'^0) &= \frac{1}{\Gamma}\Big(p'^0 - \Gamma\Big(mc\gamma(\mathbf{p}) - \beta p^x\Big)\Big) \\ &= \frac{1}{\Gamma}\Big(p'^0 - mc\gamma(\tilde{\mathbf{p}}')\Big). \end{aligned} \tag{50}$$

Using (47) and a standard property of the delta-function[1], one can write:

$$\begin{aligned} \delta \ \Big(p^0 - mc\gamma(\mathbf{p})\Big) \ \mathrm{d}p^0 &= \delta\bigg(\frac{1}{\Gamma}\Big(p'^0 - mc\gamma(\tilde{\mathbf{p}}')\Big)\bigg)\frac{\mathrm{d}p'^0}{\Gamma} \\ &= \delta\Big(p'^0 - mc\gamma(\tilde{\mathbf{p}}')\Big) \, \mathrm{d}p'^0. \end{aligned} \tag{51}$$

Some extra-care has to be used in the transformation of the integration range. The upper-limit in integral (46) corresponds, loosely speaking, to a 4-momentum with components in $(\mathcal{S})$ $(p^0 = +\infty, \mathbf{p})$; the zeroth-component of its Lorentz-transform is therefore simply $p'^0 = +\infty$. The lower-limit in integral (46) corresponds to a 4-momentum with components in $(\mathcal{S})$ $(p^0 = \epsilon(U, \mathbf{p}), \mathbf{p})$. The components of its Lorentz-transform are given by (42); let us denote them by $(\epsilon'(U, \mathbf{p}), \mathbf{p}'_\epsilon)$. The quantity $\epsilon'(U, \mathbf{p})$ is the correct lower-limit for the integral (46) using p'^0 as variable.

[1] For any non-zero real number a, $\delta(a\,x) = a^{-1}\delta(x)$.

Combining the previous results, one can write:

$$\Pi'(t',\mathbf{x}',\tilde{\mathbf{p}}') = \int_{\epsilon'(U,\mathbf{p})}^{+\infty} f(t,\mathbf{x},p^0,\mathbf{p}) \times\delta\Big(p'^0 - mc\gamma(\tilde{\mathbf{p}}')\Big)\ \mathrm{d}p'^0. \quad (52)$$

Let us now define f' by the simple relation:

$$f'(t',\mathbf{x}',p'^0,\mathbf{p}') = f(t,\mathbf{x},p^0,\mathbf{p}). \quad (53)$$

Equation (52) reads, in terms of f':

$$\Pi'(t',\mathbf{x}',\tilde{\mathbf{p}}') = \int_{\epsilon'(U,\mathbf{p})}^{+\infty} f'(t',\mathbf{x}',p'^0,\mathbf{p}') \times\delta\Big(p'^0 - mc\gamma(\tilde{\mathbf{p}}')\Big)\ \mathrm{d}p'^0. \quad (54)$$

The delta-function in the preceding equation restricts the integration to the mass-shell; since the difference between $\mathbf{p}'$ and $\tilde{\mathbf{p}}'$ precisely vanishes on the mass-shell, it follows that, in (54), $\mathbf{p}'$ can be replaced by $\tilde{\mathbf{p}}'$. This leads to:

$$\Pi'(t',\mathbf{x}',\tilde{\mathbf{p}}') = \int_{\epsilon'(U,\mathbf{p})}^{+\infty} f'(t',\mathbf{x}',p'^0,\tilde{\mathbf{p}}') \times\delta\Big(p'^0 - mc\gamma(\tilde{\mathbf{p}}')\Big)\ \mathrm{d}p'^0. \quad (55)$$

Comparison of equation (55) with definition (7), which is valid in any reference-frame, reveals that proving that f' can be used as the distribution function over the extended phase-space in $(\mathcal{S}')$ comes down to showing that the lower-limit in integral (55) can be replaced by the quantity $\epsilon(U',\tilde{\mathbf{p}}')$ defined, in accordance with (5), by:

$$\epsilon(U',\tilde{\mathbf{p}}') = \frac{\tilde{\mathbf{p}}'\cdot\mathbf{U}'}{U'^0}\cdot \quad (56)$$

Because of the presence of the "delta-function", a sufficient condition for this to be possible is simply for $\epsilon(U',\tilde{\mathbf{p}}')$ to be (strictly) inferior to $mc\gamma(\tilde{\mathbf{p}}')$. That this is indeed the case can be most easily seen by the following argument. By equation (56), one has:

$$(\epsilon(U',\tilde{\mathbf{p}}'),\tilde{\mathbf{p}}')\,U' = 0\cdot \quad (57)$$

On the other hand:

$$(mc\gamma(\tilde{\mathbf{p}}'),\tilde{\mathbf{p}}')\,U' = (mc\gamma(\mathbf{p}),\mathbf{p})\,U = mc\gamma(\mathbf{p}) > 0\cdot \quad (58)$$

Since U'^0 is a positive quantity, both preceding relations prove that $\epsilon(U',\tilde{\mathbf{p}}')$ is indeed inferior to $\gamma(\tilde{\mathbf{p}}')$. This shows that f' can be used as a distribution function on the extended phase-space in $(\mathcal{S}')$. Because of definition (53), this proves that we can treat f as a Lorentz-scalar. It surely makes no sense to state that f is (or has to be) Lorentz-invariant for Π to be a Lorentz-scalar since there is generally more than one function f which corresponds to a given physical distribution Π.

Appendix B

As explained in the Introduction and in Section 4.2, this Appendix contains some technical calculations which are necessary to prove the H-theorem but whose inclusion in the main part of this Article seemed unnecessary. More precisely, let us now present how equation (39) can be derived from equation (38).

Starting from expression (38) for the 4-divergence of the conditional entropy and making use (28) to eliminate all derivatives of f with respect to space-time coordinates, one obtains:

$$\partial_{x^\mu} S^\mu_{\mathrm{c}} = \int_{\mathcal{P}} \partial_{p^\mu}\left[I^\mu f - \partial_{p^\nu}\left(J^{\mu\nu} f\right)\right]\ln\left(\frac{f}{f^*}\right) \times\theta(p^0)\ \delta(p^2-m^2c^2)\ \mathrm{d}^4p. \quad (59)$$

We will deal with the various integrals over $\mathcal{P}$ which appear in the calculation that follows by integrating most of them by part. Border terms will then appear. Some of them trivially vanish if one supposes, as is customary in kinetic-theory and statistical physics, that phase-space distribution functions tend to zero sufficiently rapidly at infinity. One is then left with the border terms that are to be evaluated on the hyperplane $pU = 0$. These terms also vanish for the following reason. On this hyperplane, one can write:

$$p^0 = \mathbf{p}\cdot\frac{\mathbf{U}}{U^0}, \quad (60)$$

with

$$U^0 = \sqrt{1+\mathbf{U}^2}\cdot \quad (61)$$

It follows from these equations that $p^0 < \sqrt{\mathbf{p}^2}$, so that, on the hyperplane under consideration, $p^2 = (p^0)^2 - \mathbf{p}^2 < 0$. The "delta-function" which enforces the mass-shell restriction therefore vanishes on this hyperplane; this is enough to ensure that the corresponding border-terms also vanish. We will now proceed in the calculation without further mention of all border-terms.

Integrating equation (59) by part, one obtains:

$$\partial_{x^\mu} S^\mu_{\mathrm{c}} = -\int_{\mathcal{P}}\left[I^\mu f - \partial_{p^\nu}\left(J^{\mu\nu} f\right)\right]\left[\frac{\partial_{p^\mu} f}{f} - \frac{\partial_{p^\mu} f^*}{f^*}\right] \times\theta(p^0)\ \delta(p^2-m^2c^2)\ \mathrm{d}^4p$$
$$-\int_{\mathcal{P}}\left[I^\mu f - \partial_{p^\nu}\left(J^{\mu\nu} f\right)\right]\ln\left(\frac{f}{f^*}\right) \times\partial_{p^\mu}\left(\theta(p^0)\ \delta(p^2-m^2c^2)\right)\ \mathrm{d}^4p. \quad (62)$$

To ease further calculations, it is convenient to introduce the three auxiliary integrals $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$, respectively defined by:

$$\mathcal{I}_1 = -\int_{\mathcal{P}} I^\mu f\left[\frac{\partial_{p^\mu} f}{f} - \frac{\partial_{p^\mu} f^*}{f^*}\right] \times\theta(p^0)\ \delta(p^2-m^2c^2)\ \mathrm{d}^4p, \quad (63)$$

$$\mathcal{I}_2 = \int_{\mathcal{P}} \partial_{p^\nu} \left(J^{\mu\nu} f\right) \left[\frac{\partial_{p^\mu} f}{f} - \frac{\partial_{p^\mu} f^*}{f^*}\right] \times\theta(p^0)\ \delta(p^2 - m^2c^2)\ \mathrm{d}^4p \tag{64}$$

and:

$$\mathcal{I}_3 = -\int_{\mathcal{P}} \left[I^\mu f - \partial_{p^\nu}\left(J^{\mu\nu} f\right)\right] \ln\left(\frac{f}{f^*}\right) \times\partial_{p^\mu}\left(\theta(p^0)\ \delta(p^2 - m^2c^2)\right)\ \mathrm{d}^4p. \tag{65}$$

With these definitions, the 4-divergence of the entropy current simply reads:

$$\partial_{x^\mu} S^\mu_{\mathrm{c}} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \tag{66}$$

To proceed, let us concentrate first on the expression of the first integral $\mathcal{I}_1$. Because f^* is a time- and space-independent solution to Kramers equation, (28) delivers:

$$I^\mu \partial_{p^\mu} f^* = \partial_{p^\mu p^\nu}\left(\mathcal{J}^{\mu\nu}\ f^*\right) - f^* \partial_{p^\mu} I^\mu. \tag{67}$$

Inserting this relation into (63), one finds the following, alternative expression for the integral $\mathcal{I}_1$:

$$\mathcal{I}_1 = -\int_{\mathcal{P}} \left[\partial_{p^\mu}\left(I^\mu f\right) - \frac{f}{f^*}\partial_{p^\mu p^\nu}\left(J^{\mu\nu} f^*\right)\right] \times\theta(p^0)\ \delta(p^2 - m^2c^2)\ \mathrm{d}^4p. \tag{68}$$

Integrating the right hand side of this equation by part, one obtains:

$$\begin{aligned}\mathcal{I}_1 = &-\int_{\mathcal{P}} \partial_{p^\nu}\left(J^{\mu\nu} f^*\right) \partial_{p^\mu}\left(\frac{f}{f^*}\right) \times\theta(p^0)\ \delta(p^2 - m^2c^2)\ \mathrm{d}^4p \\ &+\int_{\mathcal{P}} \left[I^\mu f - \partial_{p^\nu}\left(J^{\mu\nu} f^*\right)\frac{f}{f^*}\right] \times\partial_{p^\mu}\left(\theta(p^0)\ \delta(p^2 - m^2c^2)\right)\ \mathrm{d}^4p.\end{aligned} \tag{69}$$

Introducing the tensor $\mathcal{D}$ defined by:

$$\mathcal{D}_\mu = \frac{\partial_{p^\mu} f}{f} - \frac{\partial_{p^\mu} f^*}{f^*}, \tag{70}$$

one has equivalently:

$$\begin{aligned}\mathcal{I}_1 = &-\int_{\mathcal{P}} \partial_{p^\nu}\left(J^{\mu\nu} f^*\right) \mathcal{D}_\mu\ \frac{f}{f^*} \times\theta(p^0)\ \delta(p^2 - m^2c^2)\ \mathrm{d}^4p \\ &+\int_{\mathcal{P}} \left[I^\mu f - \partial_{p^\nu}\left(J^{\mu\nu} f^*\right)\frac{f}{f^*}\right] \times\partial_{p^\mu}\left(\theta(p^0)\ \delta(p^2 - m^2c^2)\right)\ \mathrm{d}^4p.\end{aligned} \tag{71}$$

Combining (71) and (64), one obtains the following expression for the 4-divergence of the entropy current:

$$\begin{aligned}\partial_{x^\mu} S^\mu_{\mathrm{c}} = &\int_{\mathcal{P}} \left[\partial_\nu\left(J^{\mu\nu} f\right) - \frac{f}{f^*}\partial_{p\nu}\left(J^{\mu\nu} f^*\right)\right] \mathcal{D}_\mu \times\theta(p^0)\ \delta(p^2 - m^2c^2)\ \mathrm{d}^4p \\ &+\int_{\mathcal{P}} \left[I^\mu f - \partial_{p^\nu}\left(J^{\mu\nu} f^*\right)\frac{f}{f^*}\right] \times\partial_{p^\mu}\left(\theta(p^0)\ \delta(p^2 - m^2c^2)\right)\ \mathrm{d}^4p \\ &+\mathcal{I}_3.\end{aligned} \tag{72}$$

The first term on the right-hand side of (72) can be further simplified (without any additional integration by part) and, regrouping the last two integrals in (72), one has:

$$\begin{aligned}\partial_{x^\mu} S^\mu_{\mathrm{c}} = &\int_{\mathcal{P}} f J^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu\ \theta(p^0)\ \delta(p^2 - m^2c^2)\ \mathrm{d}^4p \\ &+\int_{\mathcal{P}} N^\mu \partial_{p^\mu}\left(\theta(p^0)\ \delta(p^2 - m^2c^2)\right)\ \mathrm{d}^4p\end{aligned} \tag{73}$$

where the 4-vector N is defined by:

$$\begin{aligned}N^\mu = &-\ln\left(\frac{f}{f^*}\right)\left[I^\mu f + \partial_{p^\nu}\left(J^{\mu\nu} f\right)\right] \\ &+\frac{f}{f^*}\left[I^\mu f^* - \partial_{p^\nu}\left(J^{\mu\nu} f^*\right)\right].\end{aligned} \tag{74}$$

Let I_N be the second integral in (73). We will now show that this integral vanishes. A formal blunt calculation of the derivative $\partial_{p^\mu}\left(\theta(p^0)\ \delta\left(p^2 - m^2c^2\right)\right)$ would lead to the appearance of a product of two deltas and such a product is not (mathematically) well-defined. To avoid the problem, let us introduce a class of regular functions g_ϵ, which converge uniformly towards δ when ϵ tends to zero. Instead of computing the aforementioned derivative, we will first evaluate the quantity d^μ_ϵ, defined by:

$$d^\mu_\epsilon = \partial_{p^\mu}\left(\theta(p^0)\ g_\epsilon\left(p^2 - m^2c^2\right)\right) \tag{75}$$

and, only afterwards, let ϵ tend towards zero. One has immediately:

$$\begin{aligned}d^\mu_\epsilon = &\ \delta(p^0)\ \delta^0_\mu\ g_\epsilon\left(p^2 - m^2c^2\right) \\ &+\theta(p^0)\ \partial_{p^\mu} g_\epsilon\left(p^2 - m^2c^2\right);\end{aligned} \tag{76}$$

denoting by X the variable of g_ϵ, one has therefore:

$$\begin{aligned}d^\mu_\epsilon = &\ \delta(p^0)\delta^0_\mu\ g_\epsilon\left(-\mathbf{p}^2 - m^2c^2\right) \\ &+2\ \theta(p^0)\ p^\mu\left(\frac{\mathrm{d}g_\epsilon}{\mathrm{d}X}\right)_{X=p^2-m^2c^2}.\end{aligned} \tag{77}$$

Let now ϵ tend towards zero; the function g_ϵ then tends towards δ and the first term on the right-hand side of (77) vanishes identically because the argument of this δ is strictly negative for any real momentum. One is therefore left with:

$$\lim_{\epsilon \to 0} d^{\mu}_{\epsilon} = 2\ \theta(p^0)\ p^{\mu} \delta' \left(p^2 - m^2 c^2\right). \tag{78}$$

The second integral in (73) therefore reads:

$$I_N = 2 \int_{\mathcal{P}} N^{\mu} p_{\mu} \theta(p^0)\ \delta' \left(p^2 - m^2 c^2\right) \mathrm{d}^4 p. \tag{79}$$

This integral vanishes because the scalar product of N with p vanishes identically. Indeed, there are two contributions to N. Both involve, according to (74), the differential operator $\mathcal{K}^{\mu} = I^{\mu} - \partial_{p^{\nu}} J^{\mu\nu}$. The first contribution is proportional to $\mathcal{K}^{\mu}(f)$ and the second one is proportional to $\mathcal{K}^{\mu}(f^*)$. The scalar product of p with N vanishes because the scalar product of p with $\mathcal{K}(h)$ vanishes for any (sufficiently regular) function h. Indeed, using (29) and (30):

$$\begin{aligned} p\mathcal{K}(h) = p_{\mu} &\left[\sqrt{p^2} F^{\mu}_{d} + D K^{\rho\mu\beta\nu} \partial_{p^{\nu}} \left(\frac{p_{\rho} p_{\beta}}{pU} \right) \right] h \\ &- p_{\mu}\ \partial_{p^{\nu}} \left[D K^{\rho\mu\beta\nu} \frac{p_{\rho} p_{\beta}}{pU}\ h \right]. \end{aligned} \tag{80}$$

Since the deterministic 4-force is orthogonal to p, (80) simplifies into:

$$p\mathcal{K}(h) = -D K^{\rho\mu\beta\nu}\ \frac{p_{\rho} p_{\beta} p_{\mu}}{pU}\ \partial_{p^{\nu}} h. \tag{81}$$

Starting from equation (26), a straightforward calculation shows that the contraction $K^{\rho\mu\beta\nu} p_{\rho} p_{\beta} p_{\mu}$ vanishes identically.

One is therefore left with the wanted expression:

$$\partial_{x^{\mu}} S^{\mu}_{\mathrm{c}} = \int_{\mathcal{P}} f J^{\mu\nu} \mathcal{D}_{\mu} \mathcal{D}_{\nu}\ \theta(p^0)\ \delta(p^2 - m^2 c^2)\ \mathrm{d}^4 p. \tag{82}$$

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